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HW9

MATH 527A
December 4, 2008

1. Problem 3.5.16 of Prof. Flaschka's notes.

   Let $(X, \mathcal{T})$ be a topological space, and let $A, B \subset X$. Show that $A^o \cap B^o = (A \cap B)^o$.

   Solution. First show that $A^o \cap B^o \subseteq (A \cap B)^o$. Let $x \in A^o \cap B^o$. Therefore $x \in A^o$ and $x \in B^o$. Therefore $\exists U, V$ open sets s.t. $x \in U \subset A$ and $x \in V \subset B$ So, $x \in A \cap B$ Since U, V are open, $A \cap B \supset U_1 = U \cap V \ni x$. Therefore x is an interior point of $A \cap B$. So, $A^o \cap B^o \subseteq (A \cap B)^o$.

   Now show that $(A \cap B)^o \subseteq A^o \cap B^o$. So, $\exists U \subset A \cap B$ s.t. $x \in U \implies x \in U \subset A$ and $x \in U \subset B$. So, x is an interior point of both A and B. Therefore, $(A \cap B)^o \subseteq A^o \cap B^o$.

   $\therefore A^o \cap B^o = (A \cap B)^o$.

2. Problem 3.5.22 of Prof. Flaschka's notes.

   Show that a function $f : (X, \mathcal{T}) \to (Y, \mathcal{S})$ is continuous iff for every closed subset C of Y, $f^{-1}(C)$ is a closed subset of X.

   Solution. ($\implies$) Assume $f : (X, \mathcal{T}) \to (Y, \mathcal{S})$ is continuous. Thus for $(f(x) \in V) \in \mathcal{S}$ there exists some $(U \ni x)\mathcal{T}$ s.t. $f(U) \subset V$ or equivalently $U \subset f^{-1}(V)$. This follows because $x \in U$, and $f(x) \in f(U) \subset V$ implies that $f(x) \in V$ and thus $x \in f^{-1}(V)$. Therefore $U \subset f^{-1}(V)$. Next, notice that $(f^{-1}(A))^c = \{x \in X | f(x) \notin A\} = \{x \in X | f(x) \in A^c\} = f^{-1}(A^c)$ for some subset $A \subseteq X$. So we can pull the compliment inside the inverse image so to speak. Thus given $C \subset Y$, closed, we have by definition that C^c is open and thus $C^c \in \mathcal{S}$. So given some $f(x) \in C^c$ by continuity we can find some $(U \ni x) \in \mathcal{T}$ s.t. $U \subset f^{-1}(C^c) \in \mathcal{T}$ Thus, $f^{-1}(C^c)$ is open in X and by defintion $(f^{-1}(C^c))^c = f^{-1}((C^c)^c) = f^{-1}(C)$ is closed in X.

   ($\impliedby$) Suppose $C \subset Y$ is closed implies that $f^{-1}(C) \subset X$ is closed for some function $f : (X, \mathcal{T}) \to (Y, \mathcal{S})$. Let $V \in \mathcal{S}$ be an open set. Then by definition, V^c is closed and by assumption $f^{-1}(V^c)$ is closed as well. Also by definition $f^{-1}(V^c)^c = f^{-1}(V)$ is open and thus takes open sets to open sets. Therefore f is continuous. $\therefore$ a function $f : (X, \mathcal{T}) \to (Y, \mathcal{S})$ is continuous iff for every closed subset C of Y, $f^{-1}(C)$ is a closed subset of X.

3. **Accumulation points**

   (a) Problems 3.5.26, 3.5.27, 3.5.28, 3.5.29, of Prof. Flaschka's notes.

   (3.5.26) Prove that every accumulation point of A is a point of closure of A. The converse is not true; see Example 3.5.9.

   Proof. Assume x is an accumulation point of A and take $U \ni x$ open.

$$(\implies) \quad (U \setminus \{x\}) \cap A \neq \emptyset$$
$$(\implies) \quad U \cap A \neq \emptyset.$$

But this is precisely the definition of a point of closure of A. Therefore, x is a point of closure.

$\therefore$ every accumulation point of A is a point of closure of A.

(3.5.27) Let $A = \{1, 1/2, 1/3, \dots\} \subset \mathbb{R}$ (with the usual topology). Find the points of closure of A and the accumulation points of A.

Solution.

Claim. *The points of closure of A are $A \cup \{0\}$.*

All of A are points of closure because every neighborhood of $x \in A$ intersected with A have x in the intersection. 0 is a point of closure since for all $\epsilon > 0$, $U = (-\epsilon, \epsilon)$, there exists an $N \in \mathbb{N}$ s.t. $1/n \in U$ for all $n \geq N$. We know this because $1/n \to 0$ in the metric topology. Consequently, $\bar{A} \supseteq A \cup \{0\}$.

Conversely, $(A \cup \{0\})^c = (-\infty, 0) \cup (1, \infty) \cup_{n \in \mathbb{N}} (1/(n+1), 1/n)$ is open wrt $\mathbb{R}$, so

$$\bar{A} \subseteq A \cup \{0\}.$$

Along with the earlier argument, the claim is proven.

(3.5.28) Let $(X, \mathcal{T})$ be a topological space, and let $A \subset X$ be closed. Show that every accumulation point of A is contained in A.

Solution. Try proof by contrapositive.

Assume $x \notin A$, then $x \in A^c$, which is open. Now, pick $U = A^c$. The following is true,

$$\begin{aligned} A^c \cap A &= \emptyset \\ (\implies)(U \setminus \{x\}) \cap A &= \emptyset. \end{aligned}$$

So x is not an accumulation point, since $\exists$ an open set s.t. the above is true.

$\therefore$ every accumulation point of a closed set A is in A.

(3.5.29) Let $(X, \mathcal{T})$ be a topological space, and let $A \subset X$. Show that every point of $\bar{A} \setminus A$ is an accumulation point of A.

Solution. Assume $x \in \bar{A} \setminus A$,

$$(\implies) \quad x \in \bar{A} \text{ and } x \notin A.$$

Since x is a point of closure of A any open $U \ni x$ will satisfy $U \cap A \neq \emptyset$. Additionally, since $x \notin A$, $(U \setminus \{x\}) \cap A \neq \emptyset$. This is the definition of an accumulation point of A.

$\therefore$ every point of $\bar{A} \setminus A$ is an accumulation point of A.

(b) $(X, \mathcal{T})$ is a topological space and the topology $\mathcal{T}$ is Hausdorff. $A \subseteq X$. Let A' denote the set of accumulation points of A. Show that

$$A' = \bigcap_{x \in A} \overline{A \setminus \{x\}}.$$

Solution. ($\subseteq$) First suppose $y \in A'$, in other words, y is an accumulation point of A. For all $(U \ni y)\ in \mathcal{T}$ then $U \cap A$ contains elements other than y. We can restate this definition as $U \cap (A \setminus \{y\}) \neq \emptyset$. Next I will prove the following claim;

Claim. *The number of elements in* $U \cap (A \setminus \{y\})$ *is at least two. That is,* $|U \cap (A \setminus \{y\})| \geq 2$.

Proof. By contradiction. Suppose that $U \cap (A \setminus \{y\} = \{z\}$ for some $z \in A$. The intersection cannot be the empty set because y is an accumulation point. $\mathcal{T}$ is Hausdorff, so we can find some open set $(W \ni z) \in \mathcal{T}$, $(V \ni y) \in \mathcal{T}$ s.t., $V \cap W = \emptyset$. Now consider the open set $W \cap U$, this is in $\mathcal{T}$ by the axioms of a topology, also y is in both sets thus $y \in W \cap U$. $W \cap U$ is a neighborhood of y, an accumulation point, so it should have the same properties as any other neighborhood of y. But

$$(W \cap U) \cap (A \setminus \{y\}) = W \cap (U \cap A \setminus \{y\}) = W \cap \{z\} = \emptyset.$$

This contradicts our assumption that y is an accumulation point of A, that is we have found an open set $W \cap U \ni y$ s.t. the intersection with A only contains y. Thus $|U \cap (A \setminus \{y\})| \geq 2$.

Now for $y \in A'$ we have that $U \cap (A \setminus \{x\})$ contains at least one point no matter what x is. Thus $y \in \overline{A \setminus \{x\}}$ for all $x \in A$, thus

$$A' \subseteq \bigcap_{x \in A} \overline{A \setminus \{x\}}.$$

If $y \in \bigcap_{x \in A} \overline{A \setminus \{x\}}$, then for all $(U \ni y) \in \mathcal{T}$, then $U \cap (A \setminus \{x\}) \neq \emptyset$ for all $x \in A$. Thus there exists some $z_x \in U \cap (A \setminus \{x\})$, not that $z_x \in A$ could be equal to y, but $z_x \neq x$. Then for all $(U \ni y) \in \mathcal{T}$, we have $U \cap A \setminus \{y\}$ contains z_x whenever $z_x \neq y$, and when $z_x = y$, then $y \in A$ and there exists some $z_y \in U \cap (A \setminus \{y\})$. Thus for all $(U \ni y) \in \mathcal{T}$, then $(U \cap A)$ contains at least one element not equal to y. So, $y \in A'$
$\therefore A' = \bigcap_{x \in A} \overline{A \setminus \{x\}}$.

4. **Boundary points**

(a) Problem 3.5.30 of Prof. Flaschka's notes.

Let $(X, \mathcal{T})$ be a topological space, and let $A \subset X$. There are points that "divide" A from the rest of X.

Definition. A point $x \in X$ is said to be a **boundary point** of A if $x \in \bar{A} \cap \overline{A^c}$.

Prove that x is a boundary point of A iff $x \in (A^o \cup (A^c)^o)^c$.

Proof. ($\implies$) Let x be a boundary point of A.

$$\begin{aligned}
&\implies && x \in \overline{A} \cap \overline{A^c} \\
&\implies && x \in \overline{A} \text{ and } x \in \overline{A^c} \\
&\implies && \nexists U \ni x \text{ s.t. } U \cap A = \emptyset \text{ and } \ \nexists U \ni x \text{ s.t. } U \cap A^c = \emptyset \\
&\implies && U \not\subset A \text{ and } U \not\subset A^c \\
&\implies && x \notin A^o \text{ and } x \notin (A^c)^o \\
&\implies && x \notin A^o \cup (A^c)^o \\
&\implies && x \in (A^o \cup (A^c)^o)^c.
\end{aligned}$$

($\impliedby$) Let $x \in (A^o \cup (A^c)^o)^c$.

$$\begin{aligned}
&\implies && x \notin A^o \cup (A^c)^o \\
&\implies && x \notin A^o \text{ and } x \notin (A^c)^o \\
&\implies && \nexists U \text{ open s.t. } x \in U \subset A \text{ and } x \in U \subset A^c \\
&\implies && \forall \text{ open } U \ni x \ \ U \cap A \neq \emptyset \text{ and } U \cap A^c \neq \emptyset \\
&\implies && x \in \overline{A} \text{ and } x \in \overline{A^c} \\
&\implies && x \in \overline{A} \cap \overline{A^c}
\end{aligned}$$

which implies x is a boundary point.

$\therefore$ x is a boundary point of A iff $x \in (A^o \cup (A^c)^o)^c$.

(b) Show that the definition of boundary point agrees with the definition in class, *viz.*, $x \in X$ is a boundary point of A iff for every neighborhood U of x, $U \cap A \neq \emptyset$ and $U \cap A^c \neq \emptyset$.

Solution. The definition given in class follows directly from the previous proof and the definition given for the closure of a set A being

Definition. $\{y | y \in X \text{ s.t. open sets } U \ni y \implies U \cap A \neq \emptyset\}$.

The definition for the closure of A^c follows exactly from above.

Additionally, De Morgan's laws can be used to show that $\overline{A} \cap \overline{A^c} = (A^o \cup (A^c)^o)^c$, since

$$\begin{aligned}
(A^o \cup (A^c)^o)^c &= (A^o)^c \cap ((A^c)^o)^c \\
&= \overline{A} \cap \overline{A^c}.
\end{aligned}$$

So, the proof from part (a) can be used to show consistency with the definition given in class.

5. **A topology on $\mathbb{N}$**

Let $X = \mathbb{N} \cup \{e\}$. Define a collection $\mathcal{T}$ by $A \subseteq X$ is in $\mathcal{T}$ iff A does not contain e (this includes the empty set) or $e \in A$ and A^c is finite (this includes X).

(a) Show that $\mathcal{T}$ is a topology on X.

Solution. Need to check the three properties of a topology for $\mathcal{T}$.

(i). $\emptyset \not\ni e$, so $\emptyset \in \mathcal{T}$. $e \in X$ and $X^c = \emptyset$ which is finite, so $X \in \mathcal{T}$.

(ii). Let $U, V \in \mathcal{T}$. I will break the problem up into 3 cases.

The first case is both U, V don't contain e. $U \cap V$ doesn't contain e either and therefore $U \cap V \in \mathcal{T}$.

The second case is one of U, V contains e and the other doesn't. Let $e \in U$. So, $U \cap V \not\ni e$, so $U \cap V \in \mathcal{T}$.

The third case is when both $U, V \ni e$ and U^c, V^c are finite. So, $e \in U \cap V$. Now look at $(U \cap V)^c = U^c \cup V^c$ which is finite, since both U^c, V^c are finite. So, $U \cap V \in \mathcal{T}$.

(iii). Let $\mathcal{U} \subset \mathcal{T}$, then every set U of $\mathcal{U}$ either contains e and U^c is finite, or $U \not\ni e$. Therefore, the union of all such sets can either contain e, and $(\bigcup_{U \in \mathcal{U}} U)^c = \bigcap_{U \in \mathcal{U}} U^c$ will be finite since U^c is finite. If no U contains e, then the union of all U will not contain e either. Therefore, $\bigcup_{U \in \mathcal{U}} U \in \mathcal{T}$.

$\therefore \mathcal{T}$ is a topology on X.

(b) Show that T is second countable.

Solution. I claim that the following is a base for $\mathcal{T}$: $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ where $\mathcal{B}_1 = \{\{n\} | n \in \mathbb{N}\}$ and $\mathcal{B}_2 = \{\{1, 2, 3, \ldots, N\}^c | N \in \mathbb{N}\}$. First we show that $\bigcup_{B \in \mathcal{B}} B = X$. Let x be in the union. Then trivially it is in X. Now if $y \in X$ and $y \neq e$, then $y \in \mathbb{N}$. So $y \in \{y\} \in \mathcal{B}_1$. For $e \in X$, then $e \in \{1, 2, 3, \ldots, N\}^c$ for all $N \in \mathbb{N}$. Thus

$$\bigcup_{B \in \mathcal{B}} B = X.$$

Next given $B^{(1)}$ and $B^{(2)}$ in $\mathcal{B}$, then if $n \in \mathbb{N}$ is such that $n \in B^{(1)} \cap B^{(2)}$, we have $n \in \{n\} \subseteq B^{(1)} \cap B^{(2)}$ and $\{n\} \in \mathcal{B}$. If $e \in B^{(1)} \cap B^{(2)}$, it follows that $B^{(1)} = \{1, 2, \ldots, N_1\}^c$ and $B^{(2)} = \{1, 2, \ldots, N_2\}^c$. If $N_3 = N_1 + N_2$ and $B^{(3)} = \{1, 2, \ldots, N_3\}^c$, it follows that $e \in B^{(3)} \subseteq B^{(1)} \cap B^{(2)}$ and $B^{(3)} \in \mathcal{B}$.

This proves that $\mathcal{B}$ is a base. $\mathcal{B}$ is a finite union of countable sets and thus is countable.

$\therefore \mathcal{T}$ is second countable.

(c) Show that $\mathbb{N}$ is dense in $(X, \mathcal{T})$.

Solution. I need to show that the closure of $\mathbb{N}$ is X.

First, show that $\overline{\mathbb{N}} \subseteq X$. Assume $x \in \overline{\mathbb{N}}$. Given any U open wrt X, $U \cap \mathbb{N} \neq \emptyset$. So, $U \cap \mathbb{N} \subset X$. Therefore, $x \in X$ and $\overline{\mathbb{N}} \subseteq X$.

Next, show that $X \subseteq \overline{\mathbb{N}}$. Assume $x \in X$, given any open set $U \ni x \subset X$, $U \cap X \neq \emptyset$. First, I claim that $\{e\}$ is not open. This is true since the complement of $\{e\}$ is $\mathbb{N}$, which is not finite. Therefore, since $U \neq \{e\}$, I have that $U \cap \mathbb{N} \neq \emptyset$, which implies that $x \in \overline{\mathbb{N}}$. So, $X \subseteq \overline{\mathbb{N}}$.

So, $\overline{\mathbb{N}} = X$.

$\therefore \mathbb{N}$ is dense in $(X, \mathcal{T})$.

(d) A function $f : \mathbb{N} \to \mathbb{R}$ is the same thing as a sequence x_n. We will say that $g : X \to \mathbb{R}$ is a continuous extension of f if $g(n) = f(n) \forall n \in \mathbb{N}$. Show that f has a continuous extension iff $x_n = f(n)$ is a convergent sequence. Further, the continuous extension is given by $g(e) = \lim_{n\to\infty} f(n)$.

Solution. Assume that $f(n) = x_n$ converges to a limit l. Define $g : X \to \mathbb{R}$ by $g(n) = f(n)$ for $n \in \mathbb{N}$ and $g(e) = l$. The question is whether this g is continuous (with respect to the topology $\mathcal{T}$ on X and the metric topology on $\mathbb{R}$).

Let $V \subseteq \mathbb{R}$ be open in the metric topology. If $l \notin V$, $e \notin g^{-1}(V)$ so $g^{-1}(V) \in \mathcal{T}$. If $l \in V$, then $\exists \delta > 0$ s.t. $(l - \epsilon, l + \epsilon) \subseteq V$. Further, $x_n \to l \implies \exists N \in \mathbb{N}$ s.t. $n \in \mathbb{N} \cap \{1, 2, \ldots, N\}^c$ implies $g(n) = x_n \in (l - \epsilon, l + \epsilon) \subseteq V$. Consequently, $e \in g^{-1}(V) \implies \exists B \in \mathcal{B}$ s.t. $e \in B \subseteq g^{-1}(V)$. For every $m \in \mathbb{N} \cap g^{-1}(V)$, we have $min\{m\} \subseteq g^{-1}(V)$ and $\{m\} \in \mathcal{B}$. This implies $g^{-1}(V)$ is open in $\mathcal{T}$ for all V open, so that g is continuous. This proves one half of the claim, namely that, if f is convergent, then it has a continuous extension given by $g(e) = \lim_{n\to\infty} f(n)$.

To prove the converse, we assume that f can be extended to a function $g : X \to \mathbb{R}$ such that g is continuous. From this, we want to show that $f(n) = x_n$ is a convergent sequence, and further, $\lim_{n\to\infty} f(n) = g(e)$. Let $g(e) = l$. Since g is continuous, for any given $\epsilon > 0$, we have $g^{-1}(l - \epsilon, l + \epsilon)$ is open in $\mathcal{T}$. Since $\epsilon \in g^{-1}(l - \epsilon, l + \epsilon)$, it follows that there exists a $B \in \mathcal{B}$ such that $e \in B \subseteq g^{-1}(l - \epsilon, l + \epsilon)$. Consequently, there exists $N \in \mathbb{N}$ s.t. $n > N \implies |x_n - l| < \epsilon$, and $\epsilon > 0$ was arbitrary. This proves the claim.

$\therefore$ f has a continuous extension iff $x_n = f(n)$ is a convergent sequence. Further, the continuous extension is given by $g(e) = \lim_{n\to\infty} f(n)$.

(e) Every element $a = (l, a_1, a_2, a_3, \ldots) \in \mathbb{R} \times \mathbb{R}^{\mathbb{N}}$ defines a function $f_a : X \to \mathbb{R}$ by $f_a(n) = a_n$, $f_a(e) = l$. Let $Y \subset \mathbb{R} \times \mathbb{R}^{\mathbb{N}}$ denote the set of all the convergent sequences with their associated limits, *i.e.* $(l, a_1, a_2, a_3, \ldots) \in Y \implies a_n \to l$. Find the weakest topology on X such that for all $a \in Y$, $f_a : X \to \mathbb{R}$ is continuous.

Solution. We will prove that the weak topology $\mathcal{T}_W$ generated by Y is the topology $\mathcal{T}$ from part (a).

Since T_W is the intersection of all the topologies s.t. the sequences in Y are continuous functions, it follows that $\mathcal{T}_W \subseteq \mathcal{T}$ by part (d).

Conversely, consider the sequence $y^{(n)}$ given by $y_k^{(n)} = 0$ unless $k = n$ in which case $y_n^{(n)} = 1$, with the associated limit $l = 0$. This is an element of Y. Consequently, the inverse image of $(1/2, 3/2)$ has to be open in $\mathcal{T}_W$, implying that $\forall n \in \mathbb{N}, \{n\} \in \mathcal{T}_W$. A similar argument using the sequences $z^{(n)}$ given by $z_k^{(n)} = 0$ for $k \leq n$ and $z_k^{(n)} = 1$ for $k > n$, with the the associated limit $l = 1$ shows that $\forall N \in \mathbb{N}, \{1, 2, \ldots, N\}^c \in \mathcal{T}_W$. Since every element of the base of $\mathcal{T}$ is in $\mathcal{T}_W$, it follows that every set in $\mathcal{T}$ is also in $\mathcal{T}_W$. ($\mathcal{T}_W$ is a topology so every set in $\mathcal{T}$, which is a union of sets in the base, is also a set in $\mathcal{T}_W$.)

Taken together with the earlier claim, we see that the weakest topology on X s.t. for all $a \in Y$, $f_a : X \to \mathbb{R}$ is continuous, is indeed the topology $\mathcal{T}$ from part (a).

(f) Can you find a metric on X such that the metric topology is identical to the topology $\mathcal{T}$ above? Any topology with this property is *metrizable*.

Solution. Since the metric is necessarily continuous wrt the topology, it suggests considering $\rho(x, y) = |g(x) - g(y)|$ where g is any continuous, one-to-one map from X to $\mathbb{R}$.

I'll leave you to work out the details but it is easy to show that taking $g(n) = 1/n, g(e) = 0$ in the above construction yields a metric for the topology. Show this by showing that the base for the metric topology is equivalent to the base for $\mathcal{T}$ that was constructed above.

$\therefore \mathcal{T}$ is metrizable.